

“Massive” vector field in de Sitter space

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Abstract

We present in this paper a covariant quantization of the “massive” vector field on de Sitter (dS) space based on analyticity in the complexified pseudo-Riemannian manifold. The correspondence between unitary irreducible representations of the de Sitter group and the field theory on de Sitter space-time is essential in our approach. We introduce the Wightman-Gårding axiomatic for vector field on dS space. The Hilbert space structure and the unsmeared field operators $K_\alpha(x)$ are also defined. This work is in the direct continuation of previous one concerning the scalar and the spinor cases.

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1 Introduction

In two previous papers, the various physical motivations for studying the quantum field in de Sitter space were explained [Gazeau and al, 1999-a,b]. In dS space-time, familiar Minkowskian or galilean physical quantities like mass or energy cannot be envisaged in a clear operational way. A time-like Killing vector field cannot be globally defined, and we cannot deal with a four-momentum p^μ that satisfies $p^2 = \text{constant}$. On the other hand, there is a five-component vector $(\xi^\alpha) = (\xi^0, \vec{\xi}, \xi^4)$ with $\xi^2 = 0$, which is similar to the p^μ in the null curvature limit. Again, a precise protocol of measurement of such purely de-Sitterian quantities is still lacking.

Yet, the principle of causality is well defined [Börner, Dürr, 1969; Bros, Moschella, 1996] in de Sitter space. A field called “massive” propagates inside the light-cone and corresponds to a massive Poincaré field in the null curvature limit. We call a field “massless” if it propagates on the dS light-cone and if it corresponds to a massless Poincaré field at $H = 0$. Only de Sitter vector fields of the “massive” type will be considered in this paper. In the case of a “massless” dS vector field (dS QED for instance), we have to resort to quantization *à la* Gupta-Bleuler

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in order to obtain a covariant construction. This question will be addressed in a forthcoming paper.

The field equations for the scalar, spinor, and vector fields in dS space were established by Dirac [Dirac, 1935]. The solution to the latter case was presented by Börner and Dürr [Börner, Dürr, 1969; Schomblond, Spindel, 1976] in terms of flat coordinates, which covers only one half of the dS hyperboloid. In 1986, Allen calculated the vector two-point functions in terms of the geodesic distance. The latter is independent of the choice of coordinate system [Allen, 1986]. We present in this paper the Hilbert space structure and the vector field operator in terms of coordinate-independent dS plane waves. The construction is based on analyticity properties offered by the complexified pseudo-Riemannian manifold in which dS manifold is embedded, and we refer to [Bros, Moschella, 1996] for a comprehensive review of these rigorous mathematical results concerning the functional analysis side of QFT. Our present aim is rather to make explicit the extra algebraic structure inherent to the vector case.

In Section 2 we describe the dS-vector field equation as an eigenvalue equation of the Casimir operator. We define the notations and introduce the two independent Casimir operators. In his Thesis dissertation, Takahashi [Takahashi, 1963] employed two parameters p and q for characterizing the representations of the dS group. These parameters behave like a spin (s) and a mass (m) in the Minkowskian limit. In Section 3 we solve the field equation. The solution is written in terms of a scalar field ϕ and a five-component generalized “polarization” vector \mathcal{E}

$$K(x) = \mathcal{E}(x, \xi)\phi(x).$$

In contrast to the Minkowskian situation, the vector $\mathcal{E}(x, \xi)$ is a function of the space-time point x^α . This is due to the fact that the momentum operators acquire a spin part [Börner, Dürr, 1969]. This five-component vector $\mathcal{E}(x, \xi)$ is precisely defined in order to obtain the usual polarization vector at the Minkowskian limit $H = 0$. These solutions are not globally defined due to the presence of a multiform phase factor. The solution extended to the complex dS space can actually be considered for solving this problem [Bros, Moschella, 1996].

In Section 4, we define the Wightman two-point functions ($\mathcal{W}_{\alpha\alpha'}(x, x')$), which satisfies the conditions of: a) positiveness, b) locality, c) covariance, d) normal analyticity, e) transversality and f) divergencelessness. The normal analyticity allows one to define this Wightman two-point function $\mathcal{W}_{\alpha\alpha'}(x, x')$ as the boundary value of an analytic two-point function $W_{\alpha\alpha'}(z, z')$ from the tube domains. The normal analyticity is related to the Hadamard condition which selects an unique vacuum state in dS space [Allen, 1985]. $W_{\alpha\alpha'}(z, z')$ is defined in terms of dS plane-waves in their tube domains. Then, the Hilbert space structure is introduced and the field operator $K(f)$ is defined. We also give a coordinate-independent formula for the unsmearred field operator $K(x)$.

A brief conclusion and outlook are given in Section 5. In that part, it is concluded that the “massless” vector field treatment requires an indecomposable representation of dS group and the construction of the corresponding covariant quantum field.

2 dS-vector field equation

The de Sitter space is an elementary solution of the cosmological Einstein equation. It is conveniently seen as a hyperboloid embedded in a five-dimensional Minkowski space

$$X_H = \{x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2}\}, \quad \alpha, \beta = 0, 1, 2, 3, 4, \quad (2.1)$$

where $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1)$. The kinematical group of the de Sitter space is the 10-parameter group $SO_0(1, 4)$ and its contraction limit $H = 0$ is the Poincaré group. There are two Casimir operators

$$Q^{(1)} = -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta} \\ Q^{(2)} = -W_\alpha W^\alpha, \quad W_\alpha = -\frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\eta} L^{\beta\gamma} L^{\delta\eta}, \quad (2.2)$$

where the $L_{\alpha\beta}$'s are the infinitesimal generators and $\epsilon_{\alpha\beta\gamma\delta\eta}$ is the usual antisymmetrical tensor. The de Sitter metrics reads

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = g_{\mu\nu}^{dS} dX^\mu dX^\nu, \quad \mu = 0, 1, 2, 3,$$

where the X^μ 's are the 4 space-time coordinates in dS hyperboloid. Different coordinate systems can be chosen [Mottola, 1985]. The wave equation for the vector field $A_\mu(X)$ propagating on de Sitter space can be derived from a variational principle using the action integral ($\hbar = 1$) [Allen, 1986]

$$S(A) = \int_{M_H} \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m_H^2 A^\mu A_\mu \right) d\sigma, \quad (2.3)$$

where $F^{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, m_H is a “mass”, and $d\sigma$ is the $O(1, 4)$ -invariant measure on M_H . The variational principle applied to (2.3) gives the field equation

$$\nabla_\mu F^{\mu\nu} + m_H^2 A^\nu = \nabla_\mu (\nabla^\mu A^\nu - \nabla^\nu A^\mu) + m_H^2 A^\nu = 0. \quad (2.4)$$

The antisymmetry of $F^{\mu\nu}$ implies $m_H^2 \nabla \cdot A = 0$ [Allen, 1986]. In the case of the “massive” vector field, $m_H \neq 0$ and we have

$$\nabla \cdot A = 0 \quad (2.5)$$

. Therefore the wave equation is

$$(\square_H + 3H^2 + m_H^2) A_\mu(X) = 0. \quad (2.6)$$

The five-component vector field notation $K_\alpha(x)$ is used in the following discussion. With this notation we can clarify the relation between the field and the unitary irreducible representations (UIR) of the dS group. It is also simpler to express the solution in terms of the scalar field. The four-component vector field $A_\mu(X)$ is locally determined by a five-component vector field $K_\alpha(x)$ through the relation

$$A_\mu(X) = \frac{\partial x^\alpha}{\partial X^\mu} K_\alpha(x(X)), \quad K_\alpha(x) = \frac{\partial X^\mu}{\partial x^\alpha} A_\mu(X(x)). \quad (2.7)$$

This five-component vector field quantity has to be viewed as an homogeneous function of the \mathbb{R}^5 -variables x^α with some arbitrarily chosen degree σ

$$x^\alpha \frac{\partial}{\partial x^\alpha} K_\beta(x) = x \cdot \partial K_\beta(x) = \sigma K_\beta(x). \quad (2.8)$$

It also satisfies the condition of transversality [Dirac, 1935]

$$x \cdot K(x) = 0. \quad (2.9)$$

The wave equation satisfied by K can be established in terms of the tangential (or transverse) derivative $\bar{\partial}$ on de Sitter space

$$\bar{\partial}_\alpha = \theta_{\alpha\beta} \partial^\beta = \partial_\alpha + H^2 x_\alpha x \cdot \partial, \quad x \cdot \bar{\partial} = 0, \quad (2.10)$$

where $\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta$ is the transverse projector. K corresponds to A through (2.7), so we have

$$\nabla_\mu A_\nu \longrightarrow \theta_\alpha^{\alpha'} \theta_\beta^{\beta'} \partial_{\alpha'} K_{\beta'}.$$

Hence the field equation reads:

$$(H^{-2}(\bar{\partial})^2 + 2)K(x) - 2x\bar{\partial} \cdot K(x) + H^{-2}\bar{\partial}\bar{\partial} \cdot K + H^{-2}m_H^2 K = 0, \quad (2.11)$$

which, thanks to (2.9) and divergenceless condition $\partial \cdot K = 0$, simplifies to

$$(H^{-2}(\bar{\partial})^2 + 2 + H^{-2}m_H^2)K(x) = 0. \quad (2.12)$$

In terms of Laplace-Beltrami operator on de Sitter space, $-H^2 \square_H = Q_0 = -H^2(\bar{\partial})^2$, we obtain

$$(Q_0 - 2 - H^{-2}m_H^2)K(x) = 0 = (\square_H + 2H^2 + m_H^2)K(x). \quad (2.13)$$

Let us now make the things more precise in the context of representation theory. The equation (2.13) has indeed a clear group-theoretical content. The Casimir operator $Q_1^{(1)}$ is defined by

$$Q_1^{(1)} = -\frac{1}{2}L^{\alpha\beta}L_{\alpha\beta} = -\frac{1}{2}(M^{\alpha\beta} + S^{\alpha\beta})(M_{\alpha\beta} + S_{\alpha\beta}), \quad (2.14)$$

where $M_{\alpha\beta} = -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) = -i(x_\alpha \bar{\partial}_\beta - x_\beta \bar{\partial}_\alpha)$ and the action of the spin generator $S_{\alpha\beta}$ is defined by [Gazeau, 1985]

$$S_{\alpha\beta}K_\gamma = -i(\eta_{\alpha\gamma}K_\beta - \eta_{\beta\gamma}K_\alpha). \quad (2.15)$$

The operator $Q_1^{(1)}$ commutes with the action of the group generators and consequently it is constant on each unitary irreducible representation. In fact, the vector UIR's can be classified by using the eigenvalues of $Q_1^{(1)}$, *i.e.*, $\langle Q_1^{(1)} \rangle$,

$$(Q_1^{(1)} - \langle Q_1^{(1)} \rangle)K(x) = 0. \quad (2.16)$$

From Takahashi [Takahashi, 1963] we get the following classification scheme:

$$Q^{(1)} = (-p(p+1) - (q+1)(q-2))I_d,$$

$$Q^{(2)} = (-p(p+1)q(q-1))I_d.$$

In the present context three types of vector UIR are distinguished for $SO_0(1,4)$ according to the range of values of parameters q and p . [Dixmier, 1961; Takahashi, 1963], namely
i) the UIR's $U^{1,\nu}$ of the principal series, for which

$$p = s = 1, \quad q = n + 2 = \frac{1}{2} + i\nu,$$

$$\langle Q_1^{(1)} \rangle = -n(n+3) - 1(1+1) = \nu^2 + \frac{1}{4}, \quad \nu \geq 0, \quad (2.17)$$

with parameter $\nu \in \mathbb{R}$. Note that $U^{1,\nu}$ and $U^{1,-\nu}$ are equivalent.

ii) the UIR's $V^{1,q}$ of the complementary series, for which

$$p = s = 1,$$

$$\langle Q_1^{(1)} \rangle = q - q^2 \equiv \mu, \quad 0 < \mu < \frac{1}{4}, \quad (2.18)$$

iii) the UIR's $\Pi_{p,1}^\pm$ of the discrete series, for which

$$\langle Q_1^{(1)} \rangle = -p(p+1) + 2, \quad p \geq 1, \quad q = 1. \quad (2.19)$$

In the “massless” case we have $p = q = s = 1$, $\Pi_{1,1}^\pm$.

Using (2.14) and (2.15), the action Q_1 on the five-component vector field K gives

$$Q_1 K(x) = (Q_0 - 2)K(x) + 2x\bar{\partial} \cdot K(x) - 2\partial x \cdot K(x). \quad (2.20)$$

If the vector field satisfies the divergenceless condition

$$\partial \cdot K(x) = \bar{\partial} \cdot K(x) = 0, \quad (2.21)$$

it can be biunivocally associated with a UIR of the dS group. Therefore, with the conditions $x \cdot K = 0$ and $\bar{\partial} \cdot K = 0$ and by using Eq.(2.16) and Eq.(2.20), we obtain

$$(\square_H + 2H^2 + H^2 \langle Q_1 \rangle)K_\alpha(x) = 0, \quad (2.22)$$

which has the same form as (2.13). Comparing with the latter, we get $H^2 \langle Q_1 \rangle = m_H^2$. It follows the respective mass relations for the three types of UIR:

$$m_p^2 = H^2(\nu^2 + \frac{1}{4}), \quad \nu \geq 0 \quad (\text{for the principal series}),$$

$$m_c^2 = H^2\mu, \quad 0 < \mu < \frac{1}{4} \quad (\text{for the complementary series}),$$

$$m_d^2 = H^2(2 - p(p+1)), \quad p \geq 1 \quad (\text{for the discrete series}). \quad (2.23)$$

In particular, for the discrete series representation with $p = 1$, the “mass” parameter is zero, and for $p > 1$, it is purely imaginary. We shall return to this point later. In this paper, we

only consider the “massive” vector field, *i.e.* that one for which the values assumed by the parameter m_H correspond to the principal series representations. Eq. (2.22) then reads

$$(\square_H + 2H^2 + m_p^2)K_\alpha(x) = 0. \quad (2.24)$$

Let us recall at this point the physical content of the principal series representation from the point of view of a Minkowskian observer at the limit $H = 0$. The principal series UIR $U^{1,\nu}$, $\nu \geq 0$, contracts toward the direct sum of two vector massive Poincaré UIR’s $P^<(m, 1)$ and $P^>(m, 1)$, with negative and positive energies respectively [Mickelsson, Nielderle, 1972].

$$U^{1,\nu} \xrightarrow[\nu \rightarrow \infty]{H \rightarrow 0} P^<(m, 1) \oplus P^>(m, 1). \quad (2.25)$$

The contraction limit has to be understood through the constraint $m = H\nu$. The quantity m_H , supposed to depend on H , goes to the classical mass m when the curvature goes to zero, In contrast, only one representation in the discrete series with $p = 1$ has a Minkowskian interpretation. It was denoted by $(\Pi_{1,1}^\pm)$ by Dixmier [Dixmier, 1961]. The signs \pm correspond to two types of helicity for the massless vector field. The representation $\Pi_{1,1}^+$ has a unique extension to a direct sum of two UIR’s $C(2; 1, 0)$ and $C(-2; 1, 0)$ of the conformal group $SO_0(2, 4)$ with positive and negative energies respectively [Barut, Böhm, 1970, Angelopoulos, Laoues 1998]. The latter restrict to the vector massless Poincaré RUI’s $P^>(0, 1)$ and $P^<(0, 1)$ with positive and negative energies respectively. The following diagrams illustrate these connections

$$\begin{array}{ccccc} & \mathcal{C}(2, 1, 0) & & \mathcal{C}(2, 1, 0) & \leftrightarrow \mathcal{P}^>(0, 1) \\ \Pi_{1,1}^+ \hookrightarrow & \oplus & \xrightarrow{H=0} & \oplus & \oplus \\ & \mathcal{C}(-2, 1, 0) & & \mathcal{C}(-2, 1, 0) & \leftrightarrow \mathcal{P}^<(0, 1), \end{array} \quad (2.26)$$

$$\begin{array}{ccccc} & \mathcal{C}(2, 0, 1) & & \mathcal{C}(2, 0, 1) & \leftrightarrow \mathcal{P}^>(0, -1) \\ \Pi_{1,1}^- \hookrightarrow & \oplus & \xrightarrow{H=0} & \oplus & \oplus \\ & \mathcal{C}(-2, 0, 1) & & \mathcal{C}(-2, 0, 1) & \leftrightarrow \mathcal{P}^<(0, -1), \end{array} \quad (2.27)$$

where the arrows \hookrightarrow designate unique extension and $\mathcal{P}^<^>(0, 1)$ are the massless Poincaré UIR with positive and negative energies and positive helicity. $\mathcal{P}^<^>(0, -1)$ are the massless Poincaré UIR with positive and negative energies and negative helicity. Finally, all other representations have no non-ambiguous Minkowskian counterpart.

3 dS-vector plane waves

In the five-component vector field notation $K_\alpha(x)$, the solution can be written in terms of the scalar fields. More precisely we put [Gazeau, Hans, 1988]

$$K_\alpha(x) = \bar{Z}_\alpha \phi_1 + D_{1\alpha} \phi_2, \quad (3.1)$$

where Z is a constant vector ($\bar{Z}_\alpha = \theta_{\alpha\beta} Z^\beta = Z_\alpha + H^2 x_\alpha x \cdot Z$, $x \cdot \bar{Z} = 0$) and $D_{1\alpha} = H^{-2} \bar{\partial}_\alpha$ is the generalized gradient. An arbitrary five-component vector Z_α is obtained in the same way for an arbitrary four-component spinor [Gazeau and al, 1999-b]. We choose Z_α such that at the

limit $H = 0$, one obtains the vector field in the Minkowskian space. In this limit, Z_α must be related in some sense with the usual massive polarization vectors. There are three polarization vectors for the dS vector field ($s = 1$). They generate the vector representation of the group $SU(2)$ ($2s + 1 = 3$).

Putting K_α in (2.16) and using the following relations

$$Q_1 D_1 \phi_2 = D_1 Q_0 \phi_2, \quad (3.2)$$

$$Q_1 \bar{Z}_\alpha \phi_1 = \bar{Z}_\alpha (Q_0 - 2) \phi_1 - 2H^2 D_1 (x \cdot Z) \phi_1, \quad (3.3)$$

we find that the scalar fields ϕ_1 and ϕ_2 must obey :

$$(Q_0 - (\nu^2 + \frac{9}{4})) \phi_1 = 0 = (\square_H + H^2(\nu^2 + \frac{9}{4})) \phi_1, \quad (3.4)$$

$$Q_0 \phi_2 - (\nu^2 + \frac{1}{4}) \phi_2 - 2H^2 (x \cdot Z) \phi_1 = 0. \quad (3.5)$$

So ϕ_1 is a “massive” scalar field (principal series). Now the vector field must satisfy the divergencelessness condition (2.21). Therefore we have from (3.1)

$$\partial \cdot K(x) = 0 \Rightarrow Q_0 \phi_2 = Z \cdot \bar{\partial} \phi_1 + 4H^2 Z \cdot x \phi_1.$$

We have here used the relation

$$\partial \cdot \bar{Z} \phi = Z \cdot \bar{\partial} \phi + 4H^2 Z \cdot x \phi.$$

So the field ϕ_2 can be written in terms of ϕ_1

$$\phi_2 = \frac{1}{\nu^2 + \frac{1}{4}} [Z \cdot \bar{\partial} \phi_1 + 2H^2 x \cdot Z \phi_1]. \quad (3.6)$$

Eq.(3.4) has solutions which are homogeneous with degree $\sigma = -\frac{3}{2} \pm i\nu$, and which are identified as dS plane waves [Bros, Gazeau, Moschella 1994]

$$\phi_1(x) = (Hx \cdot \xi)^\sigma, \quad (3.7)$$

where $\xi \in \mathbb{R}^5$ lies on the null cone $\mathcal{C} = \{\xi \in \mathbb{R}^5; \xi^2 = 0\}$. It follows that the two possible solutions for K are

$$\begin{aligned} K_{1\alpha}(x) &= [\bar{Z}_\alpha + \frac{1}{\nu^2 + \frac{1}{4}} D_{1\alpha} (Z \cdot \bar{\partial} + 2H^2 x \cdot Z)] (Hx \cdot \xi)^{-\frac{3}{2} + i\nu} \\ &\equiv \mathcal{E}_{1\alpha}(x, \xi, Z) (Hx \cdot \xi)^{-\frac{3}{2} + i\nu}, \end{aligned} \quad (3.8)$$

$$\begin{aligned} K_{2\alpha}(x) &= [\bar{Z}_\alpha + \frac{1}{\nu^2 + \frac{1}{4}} D_{1\alpha} (Z \cdot \bar{\partial} + 2H^2 x \cdot Z)] (Hx \cdot \xi)^{-\frac{3}{2} - i\nu} \\ &\equiv \mathcal{E}_{2\alpha}(x, \xi, Z) (Hx \cdot \xi)^{-\frac{3}{2} - i\nu}, \end{aligned} \quad (3.9)$$

where $\mathcal{E}_{1\alpha}$ are the generalized polarization vector and $\mathcal{E}_{2\alpha}^* = \mathcal{E}_{1\alpha} \equiv \mathcal{E}_\alpha$. The generalized polarization vector $\mathcal{E}_\alpha(x, \xi, Z)$ is function of the space-time point x . Its expression is given by

$$\begin{aligned} \mathcal{E}_\alpha(x, \xi, Z) = & \left(\frac{3}{2} - i\nu\right) \bar{Z}_\alpha \\ & + \frac{1}{\nu^2 + \frac{1}{4}} \left[\left(i\nu - \frac{3}{2}\right) \left(i\nu - \frac{5}{2}\right) \frac{\bar{Z} \cdot \xi}{(Hx \cdot \xi)^2} + 3 \left(i\nu - \frac{3}{2}\right) \frac{Z \cdot x}{x \cdot \xi} \right] \bar{\xi}_\alpha, \end{aligned} \quad (3.10)$$

in which $\bar{\xi}_\alpha = \theta_{\alpha\beta} \xi^\beta$. In the limit $H = 0$, $(Hx \cdot \xi)^{-\frac{3}{2}-i\nu}$ and $\mathcal{E}_\alpha(x, \xi, Z)$ behave like the plane wave $e^{ik \cdot X}$ and the polarization vector in the Minkowski space respectively. If we parametrize ξ in terms of the four-momentum of the limit Minkowskian particle of mass m ,

$$\xi = \left(\frac{k^0}{mc} = \sqrt{\frac{\vec{k}^2}{m^2 c^2} + 1}, \frac{\vec{k}}{mc}, -1 \right), \quad (3.11)$$

we have from (3.8)

$$\begin{aligned} \lim_{H \rightarrow 0} (Hx(X) \cdot \xi)^{-\frac{3}{2}+i\nu} \mathcal{E}_\alpha(x, \xi, Z) &= \left(Z_\mu - \frac{Z_\nu k^\nu}{m^2} k_\mu \right) e^{ik \cdot X} \\ &\equiv \epsilon_\mu^{(\lambda)}(k) e^{ik \cdot X}, \quad \lambda = 1, 2, 3. \end{aligned} \quad (3.12)$$

Here, the dS point $x = x_H(X)$ has been expressed in terms the Minkowskian variable $X = (X_0 = ct, \vec{X})$ measured in units of the dS radius H^{-1} :

$$\begin{aligned} x_H(X) = (x^0 = H^{-1} \sinh HX^0, \vec{x} = H^{-1} \frac{\vec{X}}{\|\vec{X}\|} \cosh HX^0 \sin H \|\vec{X}\|, \\ x^4 = H^{-1} \cosh HX^0 \cos H \|\vec{X}\|). \end{aligned} \quad (3.13)$$

Note that (X^0, \vec{X}) are global coordinates. The compact spherical nature of space at fixed X^0 is apparent in (3.13). The $\epsilon_\mu^{(\lambda)}(k)$'s are the three polarization vectors in the Minkowski space [Itzykson, Zuber, 1982]:

$$\epsilon^{(\lambda)} \cdot k = 0, \quad \epsilon^{(\lambda)} \cdot \epsilon^{(\lambda')} = \delta_{\lambda\lambda'}, \quad (3.14)$$

$$\sum_{\lambda=1}^3 \epsilon_\mu^{(\lambda)}(k) \epsilon_\nu^{(\lambda)}(k) = -(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}), \quad (3.15)$$

and $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. For simplicity we choose three five-component vectors Z^λ which obey the transverse constraints:

$$\begin{aligned} Z^{(\lambda)} \cdot \xi &= 0, \\ Z_\alpha^{(\lambda)} &= (\epsilon_\mu^{(\lambda)}(k), Z_4^{(\lambda)} = 0). \end{aligned} \quad (3.16)$$

The generalized polarization vectors now read:

$$\begin{aligned} \mathcal{E}_\alpha^{(\lambda)}(x, \xi) &\equiv \mathcal{E}_\alpha(x, \xi, Z^{(\lambda)}) \\ &= \left(\frac{3}{2} - i\nu\right) \bar{Z}_\alpha^{(\lambda)} + \left(i\nu - \frac{3}{2}\right) \left(-i\nu + \frac{1}{2}\right) \frac{Z^{(\lambda)} \cdot x}{x \cdot \xi} \bar{\xi}_\alpha. \end{aligned} \quad (3.17)$$

Finally, the two solutions for the dS-vector field take the form:

$$K_{1\alpha}(x) = \mathcal{E}_\alpha^{(\lambda)}(x, \xi)(Hx \cdot \xi)^{-\frac{3}{2}+i\nu}, \quad (3.18)$$

$$K_{2\alpha}(x) = \mathcal{E}_\alpha^{*(\lambda)}(x, \xi)(Hx \cdot \xi)^{-\frac{3}{2}-i\nu}, \quad (3.19)$$

where $\mathcal{E}_\alpha^{*(\lambda)}$ is given by (3.17). These solutions are not globally defined due to the ambiguity on the phase factor. For a complete determination, one may consider the solution in the complex de Sitter space $X_H^{(c)}$ [Bros, Moschella; 1996]

$$K_{1\alpha}(z) = \mathcal{E}_\alpha^{(\lambda)}(z, \xi)(Hz \cdot \xi)^{-\frac{3}{2}+i\nu}, \quad (3.20)$$

$$K_{2\alpha}(z) = \mathcal{E}_\alpha^{*(\lambda)}(z, \xi)(Hz \cdot \xi)^{-\frac{3}{2}-i\nu}, \quad (3.21)$$

in which $z \in X_H^{(c)} = \{z = x + iy \in \mathbb{C}^5; \eta_{\alpha\beta} z^\alpha z^\beta = (z^0)^2 - \vec{z} \cdot \vec{z} - (z^4)^2 = -H^{-2}\}$.

In the same way as in the Minkowskian space, it is seen that for the scalar and vector fields the two solutions (3.18) and (3.19) are complex conjugate of each other. On the other hand, for the spinor field there is no such relation between them [Gazeau and al 1999-b].

4 Two-point function and quantum field

We here follow the procedure already presented and discussed in previous works. Let us briefly recall the required conditions on the matrix Wightman two-point function $\mathcal{W}(x, x')$. Its matrix elements $\mathcal{W}_{\alpha\alpha'}$ are defined by

$$\mathcal{W}_{\alpha\alpha'}(x, x') = \langle \Omega, K_\alpha(x) K_{\alpha'}(x') \Omega \rangle, \quad \alpha, \alpha' = 0, \dots, 4 \quad (4.1)$$

where $x, x' \in X_H$. These functions entirely encode the theory of the generalized free fields on dS space-time X_H . They have to satisfy the following requirements:

a) Positiveness

for any test function $f_\alpha \in \mathcal{D}(X_H)$, we have

$$\int_{X_H \times X_H} f^{*\alpha}(x) \mathcal{W}_{\alpha\alpha'}(x, x') f^{\alpha'}(x') d\sigma(x) d\sigma(x') \geq 0, \quad (4.2)$$

where f^* is the complex conjugate of f and $d\sigma(x)$ denotes the dS-invariant measure on X_H [Bros, Moschella, 1996]. $\mathcal{D}(X_H)$ is the space of function C^∞ with compact support in X_H ,

b) Locality

for every space-like separated pair (x, x') , *i.e.* $x \cdot x' > -H^{-2}$,

$$\mathcal{W}_{\alpha\alpha'}(x, x') = \mathcal{W}_{\alpha'\alpha}(x', x), \quad (4.3)$$

c) Covariance

$$g^{-1} \mathcal{W}(gx, gx') g = \mathcal{W}(x, x'), \quad (4.4)$$

where $g \in SO_0(1, 4)$,

d) **Normal analyticity**

$\mathcal{W}_{\alpha\alpha'}(x, x')$ is the boundary value (in the distributional sense) of an analytic function $W_{\alpha\alpha'}(z, z')$,

e) **Transversality**

$$x \cdot \mathcal{W}(x, x') = 0 = x' \cdot \mathcal{W}(x, x'), \quad (4.5)$$

f) **Divergencelessness**

$$\partial_x \cdot \mathcal{W}(x, x') = 0 = \partial_{x'} \cdot \mathcal{W}(x, x'). \quad (4.6)$$

As it has been comprehensively justified by the theorem 4.1 of Ref. [Bros, Moschella, 1996], the analytic two-point function $W_{\alpha\alpha'}^\nu(z, z')$ is obtained from the complexified plane waves of the type (3.20) and (3.21)

$$K_1^{\xi, \lambda}(z) = (Hz \cdot \xi)^{-\frac{3}{2} + i\nu} \mathcal{E}^\lambda(z, \xi), \quad (4.7)$$

$$K_2^{\xi, \lambda}(z) = (Hz \cdot \xi)^{-\frac{3}{2} - i\nu} \mathcal{E}^{*\lambda}(z, \xi), \quad (4.8)$$

where $\mathcal{E}^\lambda(z, \xi)$ is defined by (3.17). Explicitly, it is given in terms of the following class of integral representations

$$W_{\alpha\alpha'}^\nu(z, z') = c_\nu \int_T (z \cdot \xi)^{-\frac{3}{2} - i\nu} (\xi \cdot z')^{-\frac{3}{2} + i\nu} \sum_{\lambda=1}^3 \mathcal{E}_\alpha^\lambda(z, \xi) \mathcal{E}_{\alpha'}^{*\lambda}(z', \xi) d\mu_T(\xi). \quad (4.9)$$

Here T denotes the orbital basis of $\mathcal{C}^+ = \{\xi \in \mathcal{C}; \xi^0 > 0\}$. $d\mu_T(\xi)$ is an invariant measure defined by

$$d\mu_T(\xi) = i_\Xi w_{\mathcal{C}^+} |_T, \quad (4.10)$$

where $i_\Xi w_{\mathcal{C}^+}$ denotes the 3-form on \mathcal{C}^+ obtained from the contraction of the vector field Ξ with the volume form [Bros, Moschella, 1996]

$$w_{\mathcal{C}^+} = \frac{d\xi^0 \wedge \dots \wedge d\xi^4}{d(\xi \cdot \xi)}. \quad (4.11)$$

The coefficient c_ν is a normalization constant which is fixed by local Hadamard condition. The latter selects a unique vacuum state for quantum vector fields which satisfy the dS field equation.

The functions $\mathcal{W}_{\alpha\alpha'}^\nu(x, x')$, which are solution to the wave equation (2.16), can be found simply in terms of scalar Wightman two-point functions $\mathcal{W}_i^\nu(x, x')$, $i = 1, 2$, without resorting to any explicit calculation of the integral (4.9). By using the recurrence formula (3.1) we obtain

$$\mathcal{W}_{\alpha\alpha'}^\nu(x, x') = \theta_\alpha \cdot \theta'_{\alpha'} \mathcal{W}_1^\nu(x, x') + H^{-2} \bar{\partial}_\alpha \bar{\partial}'_{\alpha'} \mathcal{W}_2^\nu(x, x'). \quad (4.12)$$

By prescribing $\mathcal{W}_{\alpha\alpha'}^\nu$ to obey Eq. (2.16) and by using the previous conditions and relations (3.2) and (3.3), it is found that \mathcal{W}_1 satisfies the equation:

$$[Q_0 - (\nu^2 + \frac{9}{4})] \mathcal{W}_1^\nu(x, x') = 0, \quad (4.13)$$

whilst \mathcal{W}_2 is given in terms of \mathcal{W}_1 by

$$\mathcal{W}_2^\nu(x, x') = \frac{1}{\nu^2 + \frac{1}{4}} [H^{-2} \bar{\partial} \cdot \bar{\partial}' \mathcal{W}_1^\nu(x, x') + 2H^2 x \cdot x' \mathcal{W}_1^\nu(x, x')]. \quad (4.14)$$

The vector Wightman function can then be written in the form:

$$\mathcal{W}_{\alpha\alpha'}^\nu(x, x') = D_{\alpha\alpha'}(x, \bar{\partial}; x', \bar{\partial}') \mathcal{W}_1^\nu(x, x'), \quad (4.15)$$

where \mathcal{W}_1^ν is solution to (4.13) and

$$D_{\alpha\alpha'} = \theta_\alpha \cdot \theta'_{\alpha'} + \frac{1}{H^2(\nu^2 + \frac{1}{4})} \bar{\partial}_\alpha \bar{\partial}'_{\alpha'} [H^{-2} \bar{\partial} \cdot \bar{\partial}' + 2H^2 x \cdot x']. \quad (4.16)$$

At the limit $H = 0$, the corresponding vector Wightman two-point function in Minkowski space is obtained in terms of the Wightman two-point function $\mathcal{W}^P(X, X')$ for the scalar field in the Minkowski space.

$$\mathcal{W}_{\mu\nu}(X, X') = \left[\eta_{\mu\nu} + \frac{1}{m^2} \frac{\partial^2}{\partial X^\mu \partial X^\nu} \right] \mathcal{W}^P(X, X'). \quad (4.17)$$

For the analytic function W_1^ν we recall that we have the following expression [Bros, Moschella, 1996]

$$W_1^\nu(z, z') = C_\nu P_{-\frac{3}{2}+i\nu}^{(5)}(H^2 z \cdot z'), \quad (4.18)$$

where $C_\nu = 2\pi^2 e^{\pi\nu} H^3 c_\nu$ and

$$c_\nu = \frac{e^{-\pi\nu} \Gamma(\frac{3}{2} + i\nu) \Gamma(\frac{3}{2} - i\nu)}{2^5 \pi^4 H}. \quad (4.19)$$

$P_{-\frac{3}{2}+i\nu}^{(5)}$ is the generalized Legendre function of the first kind. Finally we get the analytic function $W_{\alpha\alpha'}^\nu(z, z')$ in term of the latter:

$$W_{\alpha\alpha'}^\nu(z, z') = C_\nu D_{\alpha\alpha'}(z, \bar{\partial}; z', \bar{\partial}') P_{-\frac{3}{2}+i\nu}^{(5)}(H^2 z \cdot z'). \quad (4.20)$$

Its analyticity properties follow from the expression of the plane-waves (3.20) and (3.21).

The positiveness property is issued from the hermiticity condition. The proof makes use of the Fourier-Bros transformation on X_H [Bros, Moschella, 1996]. The hermiticity property is also obtained by considering boundary values of the following identity

$$W_{\alpha\alpha'}(z, z') = W_{\alpha'\alpha}^*(z'^*, z^*), \quad (4.21)$$

which is easily checked on Eq. (4.9).

The relation $g^{-1} z_1 \cdot \xi = z_1 \cdot g\xi$ and the independence of the integral (4.9) with respect to the selected orbital basis T entail the covariance property

$$g^{-1} \mathcal{W}(gx, gx') g = \mathcal{W}(x, x'). \quad (4.22)$$

In order to prove the locality condition, the following relation is needed [Bros, Moschella 1996]

$$P_{-\frac{3}{2}+i\nu}^{(5)}(H^2 z_1 \cdot z_2) = P_{-\frac{3}{2}-i\nu}^{(5)}(H^2 z_1 \cdot z_2). \quad (4.23)$$

It follows the hermiticity

$$W_{\alpha\alpha'}^\nu(z, z') = W_{\alpha'\alpha}^\nu(z'^*, z^*). \quad (4.24)$$

It is noted that the space-like separated pair (x, x') lies in the same orbit of the complex dS group as the pairs (z, z') and (z'^*, z^*) [Bros, Moschella, 1996], and so the locality condition $\mathcal{W}_{\alpha\alpha'}(x, x') = \mathcal{W}_{\alpha'\alpha}(x', x)$ holds for the former.

Now, going back to Eq. (4.9), the boundary value of $W^\nu(z, z')$ gives rise to the following integral representation of the Wightman two-point function itself:

$$\begin{aligned} \mathcal{W}_{\alpha\alpha'}^\nu(x, x') &= c_\nu \int_T [(x \cdot \xi)_+^{-\frac{3}{2}-i\nu} + e^{i\pi(-\frac{3}{2}-i\nu)} (x \cdot \xi)_-^{-\frac{3}{2}-i\nu}] \\ &[(x' \cdot \xi)_+^{-\frac{3}{2}+i\nu} + e^{-i\pi(-\frac{3}{2}+i\nu)} (x' \cdot \xi)_-^{-\frac{3}{2}+i\nu}] \sum_{\lambda=1}^3 \mathcal{E}_\alpha^\lambda(x, \xi) \mathcal{E}_{\alpha'}^{*\lambda}(x', \xi) d\mu_T. \end{aligned} \quad (4.25)$$

where $(x \cdot \xi)_+ = \begin{cases} 0 & \text{for } x \cdot \xi \leq 0 \\ (x \cdot \xi) & \text{for } x \cdot \xi > 0. \end{cases}$ [Gel'fand, Shilov, 1964]. This relation defines the two-point function in term of global plane waves on X_H .

The explicit knowledge of \mathcal{W} allows us to make the QF formalism work. The vector field $K(x)$ is expected to be an operator-valued distribution on X_H acting on a Hilbert space \mathcal{H} . In terms of Hilbert space and field-operators the properties of the Wightman two-point functions are equivalent to the following conditions [Streater, Wightman, 1964]:

1. Existence of an unitary irreducible representation of the dS group

$$U^{1,\nu}; \quad V^{1,q}; \quad \Pi_{p,1}^\pm, \quad p \neq 1,$$

2. Existence of a Hilbert space \mathcal{H}

with positive definite metric that can be described as the Hilbertian sum

$$\mathcal{H} = \mathcal{H}_0 \bigoplus \left[\bigoplus_{n=1}^{\infty} S\mathcal{H}_1^{\otimes n} \right], \quad (4.26)$$

where S denotes the symmetrization operation and $\mathcal{H}_0 = \{\lambda\Omega, \lambda \in \mathbb{C}\}$. \mathcal{H}_1 is precisely equipped with the scalar product

$$(h_1, h_2) = \int_{X_H \times X_H} h_1^{*\alpha}(x) \mathcal{W}_{\alpha\alpha'}(x, x') h_2^{\alpha'}(x') d\sigma(x) d\sigma(x') \geq 0, \quad (4.27)$$

where $h_\alpha \in \mathcal{D}(X_H)$,

3. Existence of at least one “vacuum state” Ω ,

cyclic for the polynomial algebra of field operators and invariant under the representation of dS group.

4. Covariance

of the field operators under the representation of dS group,

5. Locality

for every space-like separated pair (x, x')

$$[K_\alpha(x), K_\alpha(x')] = 0. \quad (4.28)$$

6. KMS condition or geodesic spectral condition [Bros, Moschella, 1996]

which means that the vacuum is defined as a physical state with the temperature $T = \frac{H}{2\pi}$,

7. Transversality

$$x \cdot K(x) = 0, \quad (4.29)$$

8. Divergencelessness

$$\partial \cdot K(x) = 0. \quad (4.30)$$

In terms of annihilation and creation operators, the field operator $K(f) = K^+(f) + K^-(f)$ is defined by

$$(K^-(f)h)^{(n)}(\alpha_1, x_1; \alpha_2, x_2; \dots; \alpha_n, x_n) = \sqrt{n+1} \int_{X_H \times X_H} f_\alpha(x) \mathcal{W}^{\beta\alpha}(y, x) h^{(n+1)}(\beta, y; \alpha_1, x_1; \dots; \alpha_n, x_n) d\sigma(x) d\sigma(y) \quad (4.31)$$

$$(K^+(f)h)^{(n)}(\alpha_1, x_1; \alpha_2, x_2; \dots; \alpha_n, x_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^n f_{\alpha_k}(x_k) h^{(n-1)}(\alpha_1, x_1; \dots; \hat{\alpha}_k, \hat{x}_k; \dots; \alpha_n, x_n), \quad (4.32)$$

in which symbols with hat are omitted. Using the Fourier-Bros transformation on X_H , the unsmeared operators $K(x)$ can be written as

$$K(x) = \int_T \sum_{\lambda=1}^3 \{ a_\lambda(\xi, \nu) \mathcal{E}^\lambda(x, \xi) [(x \cdot \xi)_+^{-\frac{3}{2}-i\nu} + e^{i\pi(-\frac{3}{2}-i\nu)} (x \cdot \xi)_-^{-\frac{3}{2}-i\nu}] + a_\lambda^\dagger(\xi, \nu) \mathcal{E}^{*\lambda}(x, \xi) [(x \cdot \xi)_+^{-\frac{3}{2}+i\nu} + e^{-i\pi(-\frac{3}{2}+i\nu)} (x \cdot \xi)_-^{-\frac{3}{2}+i\nu}] \} d\mu_T(\xi), \quad (4.33)$$

where $a_\lambda(\xi, \nu)$ is defined by

$$a_\lambda(\xi, \nu)|\Omega\rangle = 0, \quad (4.34)$$

The integral representation (4.33) is independent of the orbital basis T if the following relation exists

$$a_\lambda(l\xi, \nu) = l^{-\frac{3}{2}+i\nu} a_\lambda(\xi, \nu).$$

The number operator N is defined as

$$N^{(\lambda)} = \int_T d\mu_T(\xi) a_\lambda^\dagger(\xi, \nu) a_\lambda(\xi, \nu). \quad (4.35)$$

This integral is also independent of the orbital basis T . A “one-particle” state is defined via the “creation” operator in a Fock space

$$a_\lambda^\dagger(\xi, \nu)|\Omega\rangle = |\xi, \lambda\rangle. \quad (4.36)$$

So far the physical meaning of N and the states $|\xi, \lambda\rangle$ have not been clarified. Let us work with the hyperbolic-type submanifold $T_4 = T_4^+ \cup T_4^-$ defined by

$$T_4^\pm = \{\xi \in \mathcal{C}^+; \quad \xi^4 = \pm 1\}. \quad (4.37)$$

In this orbital basis we have

$$[a_\lambda(\xi, \nu), a_{\lambda'}^\dagger(\xi', \nu)] = c_\nu \delta_{\lambda\lambda'} \frac{\xi^0}{|\xi^4|} \delta^3(\vec{\xi} - \vec{\xi}'), \quad (4.38)$$

or

$$\langle \Omega, a_\lambda(\xi, \nu) a_{\lambda'}^\dagger(\xi', \nu) \Omega \rangle = c_\nu \delta_{\lambda\lambda'} \frac{\xi^0}{|\xi^4|} \delta^3(\vec{\xi} - \vec{\xi}'). \quad (4.39)$$

The relation between the quantum field in dS and its Minkowskian counterpart has now become apparent. In the limit $H = 0$, the equation (4.33) goes to the corresponding massive vector field expansion in Minkowski space-time.

5 Conclusion

In this paper, we have considered the “massive” vector field associated to the principal series of the dS group $SO_0(1, 4)$ with $\langle Q_\nu \rangle = \nu^2 + \frac{1}{4}$, $\nu \geq 0$ and with corresponding “mass” $m_p^2 = H^2(\nu^2 + \frac{1}{4})$. For the complementary series ($\langle Q_\mu \rangle = \mu$, $0 < \mu < \frac{1}{4}$) and the discrete series ($\langle Q_p \rangle = 2 - p(p + 1)$, $p \geq 1$), we can replace ν respectively by $\pm\sqrt{\mu - \frac{1}{4}}$ and $\pm\sqrt{\frac{7}{4} - p(p + 1)}$.

In the case of the complementary series the associated “mass” is positive ($m_c^2 = H^2\mu$, $0 < \mu < \frac{1}{4}$), but in the limit $H = 0$ there are no physically meaningful representation of the Poincaré group. So the physical meaning of these fields is not clear yet.

For the discrete series the associated “mass” is zero or imaginary ($m_d^2 = H^2\{2 - p(p + 1)\}$, $p \geq 1$). Only one among the discrete series representations, namely that one corresponding to $p = 1$ has a physically meaningful Poincaré limit. The latter is precisely the “massless” vector field (QED in dS space) and ν must be replaced by $\pm\frac{i}{2}$ in the previous formulas. Yet the generalized polarization vector \mathcal{E} (Eq. (3.10)) and the scalar field ϕ_2 (Eq. (3.6)) diverge at the limit. This type of singularity is actually due to the divergencelessness condition for associating this field with a specific UIR of the dS group. It can be as well understood from the equation allowing to determine ϕ_2 in terms of ϕ_1 . To solve this problem, the divergencelessness condition must be dropped out. Then the vector field is associated with an indecomposable representation of the dS group. This situation will be considered in a forthcoming paper.

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